BASIC COHOMOLOGY GROUP DECOMPOSITION OF K-CONTACT 5-MANIFOLDS

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ABSTRACT. In this paper, we consider decompositions of basic degree 2 cohomology for a compact K-contact 5-manifold (M, ξ, η, Φ, g), and conclude the pureness and fullness of Φ-invariant and Φ-anti-invariant cohomology groups. Moreover, we discuss the decomposition of the complexified basic degree 2 cohomology group. This is an analogue problem when Draghici, Li and Zhang [4] considered the C^∞ pureness and fullness of J-invariant and J-anti-invariant subgroups of the degree 2 real cohomology group $H^2(M,\mathbb{R})$ of any compact almost complex manifold (M, J).

1. Introduction

Donaldson [3] posed a question: for an almost complex structure J on a compact 4-manifold M which is tamed by a symplectic form ω, is there a symplectic form compatible with J? In order to study this question, Li and Zhang [11], Draghici, Li and Zhang [4, 5] investigated the decomposition of the real degree two de Rham cohomology group $H^2(M,\mathbb{R})$, and introduced $J-$invariant and $J-$anti-invariant subgroups $H_J^+(M)$ and $H_J^-(M)$. J is said to be C^∞ pure if $H_J^+(M) \cap H_J^-(M) = \{0\}$, C^∞ full if $H_J^+(M) + H_J^-(M) = H^2(M,\mathbb{R})$. Draghici, Li and Zhang [4] concluded that for a 4 dimensional almost complex manifold (M, J), J is C^∞ pure and full, i.e.:

$$H^2(M,\mathbb{R}) = H_J^+(M) \oplus H_J^-(M).$$

Moreover, they consider the complexified cohomology group $H^2(M,\mathbb{C}) = H^2(M,\mathbb{R}) \otimes \mathbb{C}$, and get that if J is integrable,

$$H^2(M,\mathbb{C}) = H_J^{1,1} \oplus H_J^{2,0} \oplus H_J^{0,2}.$$

For higher dimensional case, please refer to [7, 12] and references therein.

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As we all know almost complex manifolds are always of even real dimension. For odd dimensional case, we can consider contact manifolds. In this paper, we consider the decomposition of degree 2 basic cohomology group $H^2_B(\mathcal{F}_\xi)$ of a compact K-contact 5-manifold (M, ξ, η, Φ, g). There are two subgroups H^+_Φ and H^-_Φ of $H^2_B(\mathcal{F}_\xi)$ which are called Φ-invariant and Φ-anti-invariant basic cohomology group respectively. Φ is defined to be C^∞ pure if $H^+_\Phi \cap H^-_\Phi = \{0\}$, C^∞ full if $H^+_\Phi + H^-_\Phi = H^2_B(\mathcal{F}_\xi)$. We conclude that Φ is C^∞ pure and full, i.e. Theorem 2.3. Moreover, when M is Sasakian, Φ is complex C^∞ pure and full, i.e. Theorem 3.5.

2. 5 dimensional K-contact manifolds

Let us first recall some basic facts of K-contact and Sasakian manifolds. For details, please refer to [2, 8].

Suppose (M, ξ, η, Φ, g) is a $2n+1$ dimensional compact K-contact manifold, here η is the contact 1-form satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, ξ is the Reeb vector field satisfying $\eta(\xi) = 1$ and $\iota_\xi d\eta = 0$, $\Phi \in End(TM)$ such that $\Phi \circ \Phi = -id + \xi \otimes \eta$. (ξ, η, Φ) is called an almost contact structure on M. g is a Riemannian metric compatible with the almost contact structure (ξ, η, Φ) in the sense that $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$ and $g(X, \Phi Y) = d\eta(X, Y)$. The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is called K-contact if ξ is a Killing vector field of g, i.e., $L_\xi g = 0$, where L stands for the Lie derivative.

The Reeb vector field ξ is often called the characteristic vector field and uniquely determines a 1-dimensional foliation $\mathcal{F}_\xi$ on M. The line bundle L_ξ consists of tangent vectors that are tangent to the leaves of $\mathcal{F}_\xi$, and the contact subbundle D is a codimension 1 subbundle of TM whose fibers are the kernel of η. Then we have:

$$TM = L_\xi \oplus D.$$

Consider the cone on M as $C(M) = M \times \mathbb{R}^+$ with warped product metric $g_{C(M)} = \mathrm{d}r^2 + r^2 g$. Let $\Upsilon = r\frac{\partial}{\partial r}$ be the Liouville vector field. For the almost contact structure (ξ, η, Φ) on M, an almost complex structure J on $C(M)$ can be defined as a section of the endomorphism bundle of the tangent bundle $TC(M)$ satisfying:

$$JY = \Phi Y + \eta(Y)\Upsilon, J\Upsilon = -\xi.$$

(ξ, η, Φ) is said to be normal if the corresponding almost complex structure J on $C(M)$ is integrable, and a normal contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on M is called a Sasakian structure. Moreover, a pair $(M, \mathcal{S})$ is called a Sasakian manifold.

Suppose (M, ξ, η, Φ, g) is a K-contact manifold, a differential $p-$form α on M is said to be basic if

$$\iota_\xi \alpha = 0, L_\xi \alpha = 0.$$

We denote by $\Omega^p_B(\mathcal{F}_\xi)$ the basic p-forms. It is easy to check that the exterior derivation d takes basic forms to basic forms, so the subalgebra $\Omega_B(\mathcal{F}_\xi) = \oplus_p \Omega^p_B(\mathcal{F}_\xi)$ forms a subcomplex of the de Rham complex. Its cohomology ring $H^*_B(\mathcal{F}_\xi)$ is defined to be the basic cohomology ring of $\mathcal{F}_\xi$. In the following we set $d_B = d|_{\Omega_B(\mathcal{F}_\xi)}$. For any $\alpha \in \Omega^p_B(\mathcal{F}_\xi)$, the transverse Hodge star operator $\bar{*}$ can be defined as follows:

$$\bar{*}\alpha = *(\eta \wedge \alpha) = (-1)^p \iota_\xi * \alpha.$$

The adjoint operator $\delta_B : \Omega^p_B(\mathcal{F}_\xi) \to \Omega^{p-1}_B(\mathcal{F}_\xi)$ of the basic differential operator d_B:

$$\delta_B = -\bar{*} d_B \bar{*}.$$

The basic Laplacian Δ_B:

$$\Delta_B = d_B \delta_B + \delta_B d_B.$$

Analogue to the Hodge decomposition of compact Riemannian manifolds, we have the transverse Hodge decomposition [6, 10, 13]:

$$\Omega^p_B(\mathcal{F}_\xi) = \mathcal{H}^p(\mathcal{F}_\xi) \oplus \mathrm{Im}(d_B) \oplus \mathrm{Ker}(\delta_B),$$

where $\mathcal{H}^p(\mathcal{F}_\xi)$ is the space of basic harmonic p-forms defined as the kernel of

$$\Delta_B : \Omega^p_B(\mathcal{F}_\xi) \to \Omega^p_B(\mathcal{F}_\xi).$$

Specifically, for five dimensional K-contact manifold (M, ξ, η, Φ, g), one considers the contact subbundle D with bundle metric g_D induced by g. For simplicity, we still denote g_D by g. The operators

$$\frac{1}{2}(\mathrm{id} + \bar{*}), \frac{1}{2}(\mathrm{id} - \bar{*})$$

induces a decomposition of the exterior bundle Λ_D of D by decompose any α into $\dfrac{1}{2}(\alpha \pm \bar{*}\alpha)$:

$$\Lambda^2_D = \Lambda^+_g \oplus \Lambda^-_g.$$

Denote by $\Omega^\pm_g(\mathcal{F}_\xi)$ the relevant space of basic forms. Hence,

$$\Omega^2_g(\mathcal{F}_\xi) = \Omega^+_g(\mathcal{F}_\xi) \oplus \Omega^-_g(\mathcal{F}_\xi).$$

We call elements in $\Omega^+_g(\mathcal{F}_\xi)$ and $\Omega^-_g(\mathcal{F}_\xi)$ the basic self-dual and basic anti-self-dual forms. Moreover, Φ acts on the bundle of Λ^2_D by $\alpha(\cdot,\cdot) \to \alpha(\Phi\cdot, \Phi\cdot)$, so we have the splitting by decomposition $\alpha(\cdot,\cdot) = \dfrac{1}{2}[\alpha(\cdot,\cdot) \pm \alpha(\Phi\cdot, \Phi\cdot)]$:

$$\Lambda^2_D = \Lambda^+_\Phi \oplus \Lambda^-_\Phi.$$

We denote by $\Omega^+_\Phi(\mathcal{F}_\xi)$ the space of Φ-invariant basic 2-forms, $\Omega^-_\Phi(\mathcal{F}_\xi)$ the space of Φ-anti-invariant basic 2-forms. Then the Φ-invariant and Φ-anti-invariant basic cohomology groups can be defined as follows respectively:

$$H^+_\Phi(\mathcal{F}_\xi) = \{[\alpha] \in H^2_\Phi(\mathcal{F}_\xi) \mid \alpha \in \Omega^+_\Phi(\mathcal{F}_\xi)\};$$
$$H^-_\Phi(\mathcal{F}_\xi) = \{[\alpha] \in H^2_\Phi(\mathcal{F}_\xi) \mid \alpha \in \Omega^-_\Phi(\mathcal{F}_\xi)\}.$$

For a basic form α, we denote α_h, $(\alpha)^\mp_g$ and $(\alpha)^\pm_\Phi$ the relevant basic harmonic, basic (anti-)self-dual and Φ(-anti)-invariant part of α respectively.

With the notations of basic (anti-)self-dual forms, we have the following refined transverse Hodge decomposition:

Lemma 2.1. *If $\alpha \in \Omega^+_g$ and $\alpha = \alpha_h + d_B\theta + \delta_B\Psi$, then $(d_B\theta)^+_g = (\delta_B\Psi)^+_g$ and $(d_B\theta)^-_g = -(\delta_B\Psi)^-_g$. In particular,*

$$\alpha - 2(d_B\theta)^+_g = \alpha_h,$$

and $\alpha + 2(d_B\theta)^-_g = \alpha_h + 2d_B\theta$ is closed.

Proof. By the basic Hodge decomposition: $\alpha = \alpha_h + d_B\theta + \delta_B\Psi$, there holds

$$\bar{*}\alpha = \bar{*}\alpha_h + \bar{*}d_B\theta + \bar{*}\delta_B\Psi.$$

Here $\bar{*}\alpha_h$ is harmonic, since $\Delta_B\bar{*}\alpha_h = \bar{*}\Delta_B\alpha_h = 0$, and $\bar{*}\delta_B\Psi = \bar{*}(\bar{*}d_B\bar{*})\Psi = d_B\bar{*}\Psi$. Hence, $\bar{*}\delta_B\Psi = d_B\theta$, and furthermore, $\bar{*}d_B\theta = \delta_B\Psi$. Then,

$$(d_B\theta)^+_g = \frac{1}{2}(\mathrm{id} + \bar{*})(d_B\theta) = \frac{1}{2}(\mathrm{id} + \bar{*})\bar{*}(\delta_B\Psi) = (\delta_B\Psi)^+_g;$$
$$(d_B\theta)^-_g = \frac{1}{2}(\mathrm{id} - \bar{*})(d_B\theta) = \frac{1}{2}(\mathrm{id} - \bar{*})\bar{*}(\delta_B\Psi) = -(\delta_B\Psi)^-_g.$$

Therefore,

$$\begin{aligned} \alpha &= \alpha_h + (d_B\theta)^+_g + (d_B\theta)^-_g + (\delta_B\Psi)^+_g + (\delta_B\Psi)^-_g \\ &= \alpha_h + 2(d_B\theta)^+_g. \end{aligned}$$

Similarly, $\alpha + 2(d_B\theta)^-_g = \alpha_h + 2d_B\theta$. □

According to He [9], choose a coordinate $\{x, y_1, y_2, y_3, y_4\}$ such that, the frame $\{e, e_1, e_2, \Phi e_1, \Phi e_2\}$ is an adapted orthonormal frame. Its dual frame is $\{\eta, \theta_1, \theta_2, \Phi\theta_1, \Phi\theta_2\}$. Then $\omega = \frac{1}{2}d\eta = \theta_1 \wedge \Phi\theta_1 + \theta_2 \wedge \Phi\theta_2$, and $\eta \wedge (\frac{1}{2}d\eta)^2 = 2\eta \wedge \theta_1 \wedge \Phi\theta_1 \wedge \theta_2 \wedge \Phi\theta_2$ is twice volume form.

Since $\Phi\omega = \omega, \bar{*}\omega = \omega$, and we have the following equalities:

$$\begin{aligned}
\Lambda^+_\Phi &= span\{\theta_1 \wedge \Phi\theta_1, \theta_2 \wedge \Phi\theta_2, \theta_1 \wedge \theta_2 + \Phi\theta_1 \wedge \Phi\theta_2, \theta_1 \wedge \Phi\theta_2 - \Phi\theta_1 \wedge \theta_2\};\\
\Lambda^-_\Phi &= span\{\theta_1 \wedge \theta_2 - \Phi\theta_1 \wedge \Phi\theta_2, \theta_1 \wedge \Phi\theta_2 + \Phi\theta_1 \wedge \theta_2\};\\
\Lambda^+_{g_D} &= span\{\theta_1 \wedge \Phi\theta_1 - \theta_2 \wedge \Phi\theta_2, \theta_1 \wedge \theta_2 - \Phi\theta_1 \wedge \Phi\theta_2, \theta_1 \wedge \Phi\theta_2 + \Phi\theta_1 \wedge \theta_2\};\\
\Lambda^-_{g_D} &= span\{\theta_1 \wedge \Phi\theta_1 - \theta_2 \wedge \Phi\theta_2, \theta_1 \wedge \theta_2 + \Phi\theta_1 \wedge \Phi\theta_2, \theta_1 \wedge \Phi\theta_2 - \Phi\theta_1 \wedge \theta_2\},
\end{aligned}$$

there hold the following equalities:

$$\begin{aligned}
&\Lambda^+_\Phi = \mathbb{R}\omega \oplus \Lambda^-_g, \Lambda^+_g = \mathbb{R}\omega \oplus \Lambda^-_\Phi;\\
&\Lambda^+_\Phi \cap \Lambda^+_g = \mathbb{R}\omega, \Lambda^-_\Phi \cap \Lambda^-_g = 0.
\end{aligned}$$

We denote by $\mathcal{Z}^-_\Phi$ the set of closed $\Phi-$anti-invariant 2-forms, $\mathcal{H}^+_g$ the set of basic harmonic self-dual 2-forms, and $\mathcal{H}^{+,\omega^\perp}_g$ the set of basic harmonic self-dual 2-forms which are perpendicular to ω with respect to the metric induced by g. Then we have:

Lemma 2.2. *$\mathcal{Z}^-_\Phi \subset \mathcal{H}^+_g$, and $\mathcal{Z}^-_\Phi \subset H^-_\Phi$ is bijective. Furthermore, $H^-_\Phi = \mathcal{Z}^-_\Phi = \mathcal{H}^{+,\omega^\perp}_g$.*

Proof. If $\alpha \in \mathcal{Z}^-_\Phi$, then $d\alpha = 0$. Since α is self dual, i.e., $\bar{*}\alpha = \alpha$, $\delta_B\alpha = \bar{*}d_B\bar{*}\alpha = \bar{*}d_B\alpha = 0$, i.e., $\alpha \in \mathcal{H}^+_g$.
By $\Lambda^+_g = \mathbb{R}\omega \oplus \Lambda^-_\Phi$ and $\mathcal{Z}^-_\Phi = H^-_\Phi$, we have $H^-_\Phi = \mathcal{Z}^-_\Phi = \mathcal{H}^{+,\omega^\perp}_g$. □

Based on the above lemmas, we conclude the following theorem:

Theorem 2.3. *For a five dimensional closed K-contact manifold (M, ξ, η, Φ, g), Φ is C^∞ pure and full in the following sense:*

$$H^2_B(\mathcal{F}_\xi) = H^+_\Phi \oplus H^-_\Phi.$$

Proof. If $\mathfrak{a} \in H_\Phi^+ \cap H_\Phi^-$, let $\alpha^{'} \in \mathcal{Z}_\Phi^+$, $\alpha^{''} \in \mathcal{Z}_\Phi^-$ be representatives for $\mathfrak{a}$, $\alpha^{'} = \alpha^{''} + d_B\gamma$ for some basic 1-form γ. Then

$$\begin{aligned} 0 &= \int_M \alpha^{'} \wedge \alpha^{''} \wedge \eta \\ &= \int_M (\alpha^{''} + d_B\gamma) \wedge \alpha^{''} \wedge \eta \\ &= \int_M \alpha^{''} \wedge \alpha^{''} \wedge \eta + \int_M d_B\gamma \wedge \alpha^{''} \wedge \eta \\ &= \int_M \alpha^{''} \wedge \alpha^{''} \wedge \eta + \int_M \gamma \wedge d_B\alpha^{''} \wedge \eta - \int_M \gamma \wedge \alpha^{''} \wedge d_B\eta \\ &= \int_M \alpha^{''} \wedge \bar{*}\alpha^{''} \wedge \eta \\ &= \int_M |\alpha^{''}|_g^2 \, \mathrm{d}vol, \end{aligned}$$

since $\gamma \wedge \alpha^{''} \wedge d_B\eta$ is a basic 5-form, it is zero, and $\alpha^{''}$ is basic self-dual form, satisfies $\bar{*}\alpha^{''} = \alpha^{''}$.

Hence, $\alpha^{''} = 0$, i.e., $\mathfrak{a} = 0$.

Next, we prove $H^2(\mathcal{F}_\xi) = H_\Phi^+ \oplus H_\Phi^-$. Suppose the contrary, then there exists $\mathfrak{b} \in H^2(\mathcal{F}_\xi) \setminus H_\Phi^+ \oplus H_\Phi^-$. Since $H_g^- \subset H_\Phi^+$, assume $\mathfrak{b} \in H_g^+$. Let β be the basic harmonic, self-dual representative of $\mathfrak{b}$, and denote $f = \langle \beta, \omega \rangle$. Then $f \neq 0$. Otherwise, $\mathfrak{b} \in H_\Phi^-$. Consider the basic self-dual form $f\omega$. By Lemma 2.1, $(f\omega)_h + 2(f\omega)^{exact} = f\omega + 2[(f\omega)^{exact}]_g^-$ is closed and Φ-invariant. Thus, $\mathfrak{c} = [(f\omega)_h + 2(f\omega)^{exact}] \in H_\Phi^+$. However,

$$\begin{aligned} & \int \beta \wedge [(f\omega)_h + 2(f\omega)^{exact}] \wedge \eta \\ = & \int \langle \beta, (f\omega)_h + 2(f\omega)^{exact} \rangle d\mu \\ = & \int \langle \beta, f\omega + 2((f\omega)^{exact})_g^- \rangle d\mu \\ = & \int f^2 d\mu \\ \neq & \ 0. \end{aligned}$$

This contradicts the assumption that $\mathfrak{b} \perp H_\Phi^+ \oplus H_\Phi^-$.

$\square$

3. 5 dimensional Sasakian manifolds

We consider the complex basic 2-forms in this section. There holds the following decomposition:

$$\Lambda^2_{D,\mathbb{C}} = \Lambda^{2,0}_\Phi \oplus \Lambda^{1,1}_\Phi \oplus \Lambda^{0,2}_\Phi.$$

Let $\omega^i = \theta^i + \sqrt{-1}\Phi\theta^i$. Then:

$$\begin{aligned}(\Lambda^{1,1}_\Phi)_\mathbb{R} &= span\{\sqrt{-1}\omega^1 \wedge \overline{\omega}^1, \sqrt{-1}\omega^2 \wedge \overline{\omega}^2, \omega^1 \wedge \overline{\omega}^2 + \overline{\omega}^1 \wedge \omega^2, \\ &\quad \sqrt{-1}(\omega^1 \wedge \overline{\omega}^2 - \overline{\omega}^1 \wedge \omega^2)\}, \\ (\Lambda^{2,0}_\Phi \oplus \Lambda^{0,2}_\Phi)_\mathbb{R} &= span\{\omega^1 \wedge \omega^2 + \overline{\omega}^1 \wedge \overline{\omega}^2, \sqrt{-1}(\omega^1 \wedge \omega^2 - \overline{\omega}^1 \wedge \overline{\omega}^2)\}.\end{aligned}$$

By a direct calculation we have:

$$\Lambda^+_\Phi = (\Lambda^{1,1}_\Phi)_\mathbb{R}, \tag{3.1}$$

$$\Lambda^-_\Phi = (\Lambda^{2,0}_\Phi \oplus \Lambda^{0,2}_\Phi)_\mathbb{R}. \tag{3.2}$$

Definition 3.1. Let $H^{p,q}_\Phi$ be the subspace of the complexified basic cohomology $H^2_B(\mathcal{F}_\xi;\mathbb{C}) = H^2_B(\mathcal{F}_\xi;\mathbb{R}) \otimes \mathbb{C}$, consisting of classes which can be represented by a complex closed form of type (p,q).

Lemma 3.2. *There hold the following properties of the subgroups* $H^{p,q}_\Phi$*:*

$$H^{p,q}_\Phi = \overline{H^{q,p}_\Phi}; \tag{3.3}$$

$$H^{p,p}_\Phi = (H^{p,p}_\Phi \cap H^{2p}_B(\mathcal{F}_\xi;\mathbb{R})) \otimes \mathbb{C}; \tag{3.4}$$

$$(H^{p,q}_\Phi + H^{q,p}_\Phi) = ((H^{p,q}_\Phi + H^{q,p}_\Phi) \cap H^{p+q}_B(\mathcal{F}_\xi;\mathbb{R})) \otimes \mathbb{C}. \tag{3.5}$$

Proof. Choose a complex form Ψ, then (3.3) follows from the fact that Ψ is closed if and only if its conjugate $\overline{\Psi}$ is closed. (3.4) and (3.5) follow from a fact in linear algebra:

Let V be a real vector space and W a complex subspace of $V \otimes_\mathbb{R} \mathbb{C}$, and W is invariant under conjugation as a subspace. Then $W = (W \cap V) \otimes \mathbb{C}$. See [1]. □

Lemma 3.3. *For a compact 5-dimensional K-contact manifold, there hold the following:*

$$H^+_\Phi = H^{1,1}_\Phi \cap H^2(\mathcal{F}_\xi;\mathbb{R}); \tag{3.6}$$

$$H^{1,1}_\Phi = H^+_\Phi \otimes_\mathbb{R} \mathbb{C}. \tag{3.7}$$

Proof. We first prove (3.6). By (3.2) we have $H^+_\Phi \subseteq H^{1,1}_\Phi \cap H^2(\mathcal{F}_\xi;\mathbb{R})$. For the converse inclusion, we choose an element $[\rho] \in H^{1,1}_\Phi \cap H^2(\mathcal{F}_\xi;\mathbb{R})$, such that ρ is a d_B closed basic $(1,1)$ form of the form

$\rho = \sigma + d_B\tau$, where σ a d_B closed basic real form. Hence, $[\rho]$ is also represented by the real d_B closed basic $(1,1)$ form $\frac{1}{2}(\rho+\overline{\rho}) = \sigma + d_B(\frac{\tau+\overline{\tau}}{2})$. This shows that $H^+_\Phi \supseteq H^{1,1}_\Phi \cap H^2(\mathcal{F}_\xi;\mathbb{R})$.

The relation (3.7) is a direct consequence of (3.4) with $p = 1$ and (3.6). □

Lemma 3.4. *Suppose that M is a compact 5-dimensional K-contact manifold. If the contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is normal, i.e., $(M, \mathcal{S})$ is Sasakian, there hold the following:*

$$(H^{2,0}_\Phi + H^{0,2}_\Phi) = H^-_\Phi \otimes_\mathbb{R} \mathbb{C}; \tag{3.8}$$

$$H^-_\Phi = (H^{2,0}_\Phi + H^{0,2}_\Phi) \cap H^2(\mathcal{F}_\xi;\mathbb{R}). \tag{3.9}$$

Proof. Choose a complex form $\Theta = \alpha + i\Phi\alpha \in \Omega^{2,0}_\Phi$, where $\alpha \in \Omega^-_\Phi$. Since $d_B = \partial_B + \overline{\partial}_B$ and $2\alpha = \Theta + \overline{\Theta}$, we have

$$2d_B\alpha = (\partial_B + \overline{\partial}_B)(\Theta + \overline{\Theta}) = \partial_B\overline{\Theta} + \overline{\partial}_B\Theta.$$

Here we have used the fact that $\partial_B\Theta = 0 = \overline{\partial}_B\overline{\Theta}$, since M is 5-dimensional and $\partial_B\Theta$ is a basic $(3,0)$ form, $\overline{\partial}_B\overline{\Theta}$ is a basic $(0,3)$ form. Therefore,

$$d_B\alpha = 0 \Leftrightarrow \partial_B\overline{\Theta} = 0 \Leftrightarrow \overline{\partial}_B\Theta = 0.$$

Similarly,

$$d_B(\Phi\alpha) = 0 \Leftrightarrow \partial_B(\overline{i\Theta}) = 0 \Leftrightarrow \overline{\partial}_B(i\Theta) = 0.$$

Moreover, $\overline{\partial}_B\Theta = 0$ if and only if $\overline{\partial}_B(i\Theta) = 0$. Then it follows that $d_B\alpha = 0$ if and only if $d_B(\Phi\alpha) = 0$. Therefore, $(H^{2,0}_\Phi + H^{0,2}_\Phi) = H^-_\Phi \otimes_\mathbb{R} \mathbb{C}$.

The relation (3.9) follows from (3.5) with $(p,q) = (2,0)$ and (3.8). □

Next we suppose the contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$. Combining with Lemma 3.3 and Lemma 3.4, there holds the following:

Theorem 3.5. *For a compact 5-dimensional K-contact manifold $(M, \mathcal{S})$, Φ is always complex C^∞ pure in the sense:*

$$H^{1,1}_\Phi \cap H^{2,0}_\Phi \cap H^{0,2}_\Phi = \{0\}.$$

Moreover, if $\mathcal{S}$ is normal, then Φ is also complex C^∞ full in the sense:

$$H^2(\mathcal{F}_\xi;\mathbb{C}) = H^{1,1}_\Phi \oplus H^{2,0}_\Phi \oplus H^{0,2}_\Phi$$

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